

# Level-rank duality of untwisted and twisted D-branes of the $\widehat{\mathfrak{so}}(N)_K$ WZW model<sup>1</sup>

Stephen G. Naculich<sup>2</sup> and Benjamin H. Ripman

*Department of Physics  
Bowdoin College  
Brunswick, ME 04011*

## Abstract

We analyze the level-rank duality of untwisted and  $\varepsilon$ -twisted D-branes of the  $\widehat{\mathfrak{so}}(N)_K$  WZW model. Untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  are characterized by integrable tensor and spinor representations of  $\widehat{\mathfrak{so}}(N)_K$ . Level-rank duality maps untwisted  $\widehat{\mathfrak{so}}(N)_K$  D-branes corresponding to (equivalence classes of) tensor representations onto those of  $\widehat{\mathfrak{so}}(K)_N$ . The  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  are characterized by (a subset of) integrable tensor and spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$ . Level-rank duality maps spinor  $\varepsilon$ -twisted  $\widehat{\mathfrak{so}}(2n)_{2k}$  D-branes onto those of  $\widehat{\mathfrak{so}}(2k)_{2n}$ . For both untwisted and  $\varepsilon$ -twisted D-branes, we prove that the spectrum of an open string ending on these D-branes is isomorphic to the spectrum of an open string ending on the level-rank-dual D-branes.

---

<sup>1</sup>Research supported in part by the NSF under grant PHY-0456944

<sup>2</sup> naculich@bowdoin.edu

# 1 Introduction

It has long been known that the modular transformation matrix and fusion algebra of the Wess-Zumino-Witten (WZW) model with affine Lie algebra  $\widehat{\mathfrak{su}}(N)_K$  are closely related to those of the WZW model with affine Lie algebra  $\widehat{\mathfrak{su}}(K)_N$  (level-rank duality) [1, 2, 3]. Similar dualities have been shown for WZW models with affine Lie algebras related by  $\widehat{\mathfrak{sp}}(n)_k \leftrightarrow \widehat{\mathfrak{sp}}(k)_n$  and  $\widehat{\mathfrak{so}}(N)_K \leftrightarrow \widehat{\mathfrak{so}}(K)_N$  [2, 3], and also for  $\widehat{\mathfrak{u}}(N)_{K,N(K+N)} \leftrightarrow \widehat{\mathfrak{u}}(K)_{N,K(K+N)}$  [4].

More recently, it has been shown [5, 6, 7] that the untwisted and twisted D-branes in the boundary  $\widehat{\mathfrak{su}}(N)_K$  WZW model [8]–[25] respect level-rank duality; that is, there exists a one-to-one map between the (equivalence classes of) D-branes of  $\widehat{\mathfrak{su}}(N)_K$  and those of  $\widehat{\mathfrak{su}}(K)_N$ . The open-string spectra associated with level-rank-dual D-branes are isomorphic, and the charges of level-rank-dual untwisted D-branes are equal (modulo sign), with a slightly more complicated relationship holding between the charges of twisted D-branes. Level-rank duality also holds for the D-branes of  $\widehat{\mathfrak{sp}}(n)_k$  [6].

In this paper, we continue the story by establishing the level-rank duality of the untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  and of the twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$ . In this case, level-rank duality is partial and holds only for a subset of the D-branes of the theory. Moreover, in this case we find no simple relation between the charges of level-rank-dual D-branes.

We begin by summarizing our results. Untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  correspond to untwisted Cardy states  $|\alpha\rangle\rangle_C$  (boundary states of the bulk WZW model), which are labelled by integrable highest-weight representations  $\alpha$  (both tensors and spinors) of the untwisted affine Lie algebra  $\widehat{\mathfrak{so}}(N)_K$ . Only untwisted *tensor* D-branes exhibit level-rank duality,<sup>3</sup> and the duality is one-to-one between equivalence classes  $[\alpha]$  of integrable tensor representations generated by the  $\mathbb{Z}_2$ -automorphisms  $\sigma$  and (when  $N$  is even)  $\varepsilon$  of the  $\widehat{\mathfrak{so}}(N)_K$  algebra. (We denote by  $\sigma$  the simple current symmetry of  $\widehat{\mathfrak{so}}(N)_K$  that acts on the Dynkin indices of a representation by<sup>4</sup>  $a_0 \leftrightarrow a_1$ . We denote by  $\varepsilon$  the “chirality-flip” symmetry of  $\widehat{\mathfrak{so}}(2n)_K$  that acts on the Dynkin indices of a representation by  $a_n \leftrightarrow a_{n-1}$ . For  $\widehat{\mathfrak{so}}(2n+1)_K$ , we define  $\varepsilon$  to be the identity.) The boundary state corresponding to the equivalence class  $[\alpha]$  may be written as

$$|[\alpha]\rangle\rangle = \frac{1}{\sqrt{2}^{t(\alpha)-s(\alpha)+3}} \left[ |\alpha\rangle\rangle_C + |\sigma(\alpha)\rangle\rangle_C + |\varepsilon(\alpha)\rangle\rangle_C + |\varepsilon(\sigma(\alpha))\rangle\rangle_C \right] \quad (1.1)$$

where

$$s(\alpha) = \begin{cases} 1 & \text{if } \alpha \neq \varepsilon(\alpha), \\ 0 & \text{if } \alpha = \varepsilon(\alpha), \end{cases} \quad t(\alpha) = \begin{cases} 1 & \text{if } \alpha = \sigma(\alpha), \\ 0 & \text{if } \alpha \neq \sigma(\alpha). \end{cases} \quad (1.2)$$

Equivalence classes  $[\alpha]$  of integrable tensor representations of  $\widehat{\mathfrak{so}}(N)_K$  are characterized by Young tableaux with  $\leq N/2$  rows and  $\leq K/2$  columns. Level-rank duality acts by transposing these tableaux, inducing a one-to-one correspondence  $[\alpha] \rightarrow [\tilde{\alpha}]$  between equivalence classes of  $\widehat{\mathfrak{so}}(N)_K$  and  $\widehat{\mathfrak{so}}(K)_N$ , and therefore between the untwisted D-branes that correspond to the boundary states (1.1). We show that the spectrum of representations carried by an open string stretched between untwisted  $\widehat{\mathfrak{so}}(N)_K$  D-branes corresponding to  $[\alpha]$  and  $[\beta]$  is

---

<sup>3</sup>Except for  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ , where a level-rank map can also be defined between equivalence classes of untwisted spinor D-branes.

<sup>4</sup>Except for  $\widehat{\mathfrak{so}}(4)_K$ , in which case  $\sigma$  acts by  $a_0 \leftrightarrow \min(a_1, a_2)$ .

isomorphic to that carried by an open string stretched between untwisted  $\widehat{\mathfrak{so}}(K)_N$  D-branes corresponding to  $[\tilde{\alpha}]$  and  $[\tilde{\beta}]$ .

The  $\widehat{\mathfrak{so}}(2n)_K$  WZW model contains, in addition to untwisted D-branes, a class of D-branes twisted by the symmetry  $\varepsilon$ . These  $\varepsilon$ -twisted D-branes can be characterized [20] by (a subset of) the integrable highest-weight representations (both tensors and spinors) of the untwisted affine Lie algebra  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ . Only *spinor*  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  exhibit level-rank duality, which involves a one-to-one map  $\alpha \rightarrow \hat{\alpha}$  between the spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  and the spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2k)_{2n}$ . We show that the spectrum of representations carried by an open string stretched between  $\varepsilon$ -twisted  $\widehat{\mathfrak{so}}(2n)_{2k}$  D-branes corresponding to  $\alpha$  and  $\beta$  is isomorphic to that carried by an open string stretched between  $\varepsilon$ -twisted  $\widehat{\mathfrak{so}}(2k)_{2n}$  D-branes corresponding to  $\hat{\alpha}$  and  $\hat{\beta}$ .

This paper is organized as follows. Section 2 briefly reviews the Ishibashi and Cardy states of the WZW model, and in sec. 3, we characterize the integrable highest-weight representations of  $\widehat{\mathfrak{so}}(N)_K$ . Section 4 describes the level-rank duality of the (equivalence classes of) untwisted D-branes corresponding to tensor representations of  $\widehat{\mathfrak{so}}(N)_K$  and to spinor representations of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ . The  $\varepsilon$ -twisted Ishibashi and Cardy states of  $\widehat{\mathfrak{so}}(2n)_K$  are reviewed in sec. 5, and in sec. 6 we describe the level-rank duality of spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$ .

## 2 Untwisted and twisted D-branes of WZW models

In this section, we briefly review some general aspects of untwisted and twisted D-branes of the WZW model and their relation to the Cardy and Ishibashi states of the closed-string sector, drawing on refs. [9, 10, 18, 20].

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is an untwisted affine Lie algebra  $\hat{g}_K$  at level  $K$ . We only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}^{\text{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \bar{V}_{\lambda^*} \quad (2.1)$$

where  $V$  and  $\bar{V}$  represent left- and right-moving states respectively,  $\lambda^*$  denotes the representation conjugate to  $\lambda$ , and  $P_+^K$  is the set of integrable highest-weight representations of  $\hat{g}_K$ .

D-branes of the WZW model may be described algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We only consider boundary conditions on the currents of the affine Lie algebra of the form

$$\left[ J^a(z) - \omega \bar{J}^a(\bar{z}) \right] \Big|_{z=\bar{z}} = 0 \quad (2.2)$$

where  $\omega$  is an automorphism of the Lie algebra  $g$ . These boundary conditions leave unbroken the  $\hat{g}_K$  symmetry, as well as the conformal symmetry, of the theory.

### Twisted Ishibashi states.

Open-closed string duality allows one to correlate the boundary conditions (2.2) of the boundary WZW model with coherent states  $|B\rangle\rangle^\omega \in \mathcal{H}^{\text{closed}}$  of the bulk WZW model satisfying

$$\left[ J_m^a + \omega \bar{J}_{-m}^a \right] |B\rangle\rangle^\omega = 0, \quad m \in \mathbb{Z} \quad (2.3)$$

where  $J_m^a$  are the modes of the affine Lie algebra generators. Solutions of eq. (2.3) that belong to a single sector  $V_\mu \otimes \bar{V}_{\omega(\mu)^*}$  of the bulk WZW model are known as  $\omega$ -twisted Ishibashi states  $|\mu\rangle\rangle_I^\omega$  [26]. As we are considering the diagonal closed-string theory (2.1),  $\omega$ -twisted Ishibashi states only exist when  $\mu = \omega(\mu)$ , and so are labelled by  $\mu \in \mathcal{E}^\omega$ , where  $\mathcal{E}^\omega$  is the subset of integrable representations of  $\hat{g}_K$  that satisfy  $\omega(\mu) = \mu$ . Equivalently,  $\mu$  corresponds to an integrable representation of  $\check{g}^\omega$ , the orbit Lie algebra associated with  $\hat{g}_K$  [27].

### Twisted Cardy states.

A coherent state  $|B\rangle\rangle^\omega$  that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions [28]. Solutions of eq. (2.3) that satisfy the Cardy conditions are denoted  $\omega$ -twisted Cardy states  $|\alpha\rangle\rangle_C^\omega$ , where the labels  $\alpha$  take values in  $\mathcal{B}^\omega$ , the set of integrable representations of the  $\omega$ -twisted affine Lie algebra  $\hat{g}_K^\omega$  [10]. The  $\omega$ -twisted Cardy states may be expressed as linear combinations of  $\omega$ -twisted Ishibashi states

$$|\alpha\rangle\rangle_C^\omega = \sum_{\mu \in \mathcal{E}^\omega} \frac{\psi_{\alpha\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\rangle_I^\omega \quad (2.4)$$

where  $S_{\lambda\mu}$  is the modular transformation matrix of  $\hat{g}_K$ , 0 denotes the identity representation, and the coefficients  $\psi_{\alpha\mu}$  may be identified with the modular transformation matrices of the  $\omega$ -twisted affine Lie algebra  $\hat{g}_K^\omega$  [10].

The  $\omega$ -twisted D-branes of  $\hat{g}_K$  correspond to the  $\omega$ -twisted Cardy states  $|\alpha\rangle\rangle_C^\omega$  and are therefore also labelled by  $\alpha \in \mathcal{B}^\omega$ . The spectrum of an open string stretched between  $\omega$ -twisted D-branes labelled by  $\alpha$  and  $\beta$  is encoded in the open-string partition function

$$Z_{\alpha\beta}^{\text{open}}(\tau) = \sum_{\lambda \in P_+^K} n_{\beta\lambda}^\alpha \chi_\lambda(\tau) \quad (2.5)$$

where  $\chi_\lambda(\tau)$  is the affine character of the integrable highest-weight representation  $\lambda$  of  $\hat{g}_K$ . The multiplicity  $n_{\beta\lambda}^\alpha$  of the representation  $\lambda$  carried by the open string may be expressed as [20]

$$n_{\beta\lambda}^\alpha = \sum_{\mu \in \mathcal{E}^\omega} \frac{\psi_{\alpha\mu}^* S_{\lambda\mu} \psi_{\beta\mu}}{S_{0\mu}}. \quad (2.6)$$

### Untwisted Ishibashi and Cardy states.

Untwisted Cardy states  $|\alpha\rangle\rangle_C$  and untwisted Ishibashi states  $|\mu\rangle\rangle_I$  are solutions of eq. (2.3) with  $\omega = 1$ , and both are labelled by integrable representations of  $\hat{g}_K$ . The matrix  $\psi_{\alpha\mu}$  in eq. (2.4) relating the untwisted Cardy states to the untwisted Ishibashi states is given by

the modular transformation matrix  $S_{\alpha\mu}$  of  $\hat{g}_K$  [28]. Consequently, by virtue of eq. (2.6) and the Verlinde formula for the fusion coefficients [29]

$$n_{\beta\lambda}{}^\alpha = \sum_{\mu \in P_+^K} \frac{S_{\beta\mu} S_{\lambda\mu} S_{\alpha\mu}^*}{S_{0\mu}} = N_{\beta\lambda}{}^\alpha \quad (2.7)$$

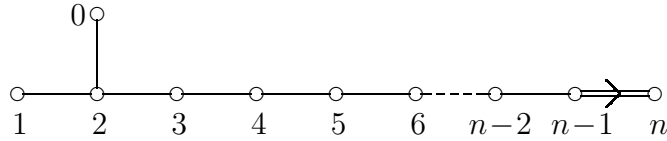
the multiplicities  $n_{\beta\lambda}{}^\alpha$  of the representations carried by an open string stretched between two untwisted D-branes  $\alpha$  and  $\beta$  are given by the fusion coefficients  $N_{\beta\lambda}{}^\alpha$  of the WZW model.

### 3 Integrable representations of $\widehat{\mathfrak{so}}(N)_K$

In this section, we review some details about the integrable representations of  $\widehat{\mathfrak{so}}(N)_K$  used throughout this paper.<sup>5</sup> Integrable representations of an affine Lie algebra  $\hat{g}_K$  have non-negative Dynkin indices  $(a_0, a_1, \dots, a_r)$  that satisfy  $\sum_{i=0}^r m_i a_i = K$ , where  $m_i$  are the dual Coxeter labels of the Dynkin diagram for  $\hat{g}_K$ , and  $r+1$  is the rank of  $\hat{g}_K$ .

**Integrable representations of  $\widehat{\mathfrak{so}}(2n+1)_K$ .**

The Dynkin diagram for  $\widehat{\mathfrak{so}}(2n+1)_K$  is



and the dual Coxeter labels are  $(m_0, m_1, m_2, \dots, m_{n-1}, m_n) = (1, 1, 2, \dots, 2, 1)$ , where the labelling of nodes is indicated on the diagram. Integrable representations of  $\widehat{\mathfrak{so}}(2n+1)_K$  thus have Dynkin indices that satisfy<sup>6</sup>

$$a_0 + a_1 + 2(a_2 + \dots + a_{n-1}) + a_n = K. \quad (3.1)$$

An even or odd value of  $a_n$  corresponds, respectively, to a tensor or spinor representation of  $\mathfrak{so}(2n+1)$ . With each irreducible tensor representation of  $\mathfrak{so}(2n+1)$  may be associated a Young tableau whose row lengths  $\ell_i$  are given by

$$\ell_i = \begin{cases} \frac{1}{2}a_n + \sum_{j=i}^{n-1} a_j & \text{for } 1 \leq i \leq n-1, \\ \frac{1}{2}a_n & \text{for } i = n. \end{cases} \quad (3.2)$$

The integrability condition (3.1) is equivalent to the constraint  $\ell_1 + \ell_2 \leq K$  on the row lengths of the tableau.

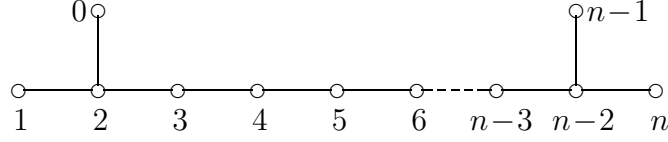
We may also formally use eq. (3.2) to define row lengths for a spinor representation. These row lengths are all half-integers, and correspond to a “Young tableau” containing a column of “half-boxes.”

<sup>5</sup>Throughout this paper,  $N \geq 3$  is understood.

<sup>6</sup> Throughout this paper, by  $\widehat{\mathfrak{so}}(3)_K$  we mean the affine Lie algebra  $\widehat{\mathfrak{su}}(2)_{2K}$ . Its integrable representations have  $\mathfrak{so}(3)$  Young tableaux that obey  $\ell_1 \leq K$ . Since  $\ell_1 = \frac{1}{2}a_1$ , this means that eq. (3.1) is replaced by  $a_0 + a_1 = 2K$  when  $n = 1$ .

## Integrable representations of $\widehat{\mathfrak{so}}(2n)_K$ .

The Dynkin diagram for  $\widehat{\mathfrak{so}}(2n)_K$  is



and the dual Coxeter labels are  $(m_0, m_1, m_2, \dots, m_{n-2}, m_{n-1}, m_n) = (1, 1, 2, \dots, 2, 1, 1)$ , where the labelling of nodes is indicated on the diagram. Integrable representations of  $\widehat{\mathfrak{so}}(2n)_K$  thus have Dynkin indices that satisfy<sup>7</sup>

$$a_0 + a_1 + 2(a_2 + \dots + a_{n-2}) + a_{n-1} + a_n = K. \quad (3.3)$$

An even or odd value of  $a_n - a_{n-1}$  corresponds, respectively, to a tensor or spinor representation of  $\mathfrak{so}(2n)$ .

The Dynkin diagram of  $\mathfrak{so}(2n)$  (and also  $\widehat{\mathfrak{so}}(2n)_K$ ) is invariant under the exchange of the  $(n-1)^{\text{th}}$  and  $n^{\text{th}}$  nodes. This gives rise to a  $\mathbb{Z}_2$ -automorphism  $\varepsilon$  of the  $\mathfrak{so}(2n)$  Lie algebra, which exchanges representations with Dynkin indices  $(\dots, a_{n-1}, a_n)$  and  $(\dots, a_n, a_{n-1})$ . This automorphism may be dubbed [20] “chirality flip” as it exchanges the two fundamental spinor representations of opposite chirality.

For each representation of  $\mathfrak{so}(2n)$  we may define

$$\ell_i = \begin{cases} \frac{1}{2}(a_n + a_{n-1}) + \sum_{j=i}^{n-2} a_j & \text{for } 1 \leq i \leq n-2, \\ \frac{1}{2}(a_n + a_{n-1}) & \text{for } i = n-1, \\ \frac{1}{2}(a_n - a_{n-1}) & \text{for } i = n. \end{cases} \quad (3.4)$$

in terms of which the integrability condition (3.3) becomes  $\ell_1 + \ell_2 \leq K$ . The *absolute values* of  $\ell_i$  represent the row lengths of a Young tableau  $A$  with up to  $n$  rows. (For spinor representations, these row lengths are all half-integer, and correspond to a Young tableau containing a column of half-boxes.) When  $a_n = a_{n-1}$ , the Young tableau has  $n-1$  or fewer rows, and corresponds to a unique irreducible  $\mathfrak{so}(2n)$  representation  $a$ , one which is invariant under  $\varepsilon$ . When  $a_n \neq a_{n-1}$ , the Young tableau has precisely  $n$  rows and corresponds to two distinct representations,  $a$  and  $\varepsilon(a)$ . Hence we may consider the Young tableau  $A$  as labelling either an irreducible ( $a$ ) or a reducible ( $a \oplus \varepsilon(a)$ ) representation of  $\mathfrak{so}(2n)$ , depending respectively on whether the representation  $a$  is or is not invariant under  $\varepsilon$ . We thus write

$$A = 2^{s(a)-1} [a \oplus \varepsilon(a)] \quad (3.5)$$

where

$$s(a) = \begin{cases} 1 & \text{if } \ell_n \neq 0; \text{ that is, } a \neq \varepsilon(a), \\ 0 & \text{if } \ell_n = 0; \text{ that is, } a = \varepsilon(a). \end{cases} \quad (3.6)$$

(For *all* representations of  $\mathfrak{so}(2n+1)$ , we define  $\varepsilon(a) = a$  and  $s(a) = 0$ .)

---

<sup>7</sup> For  $\widehat{\mathfrak{so}}(4)_K$ , we take the integrability condition to be  $a_0 + \max(a_1, a_2) = K$ , which is equivalent to  $\ell_1 + |\ell_2| \leq K$ .

Let  $S_{ab}$  denote the (symmetric) modular transformation matrix of  $\widehat{\mathfrak{so}}(N)_K$ , an explicit formula for which may be found, for example, in ref. [3]. We define

$$\begin{aligned} S_{Ab} &= 2^{s(a)-1} [S_{ab} + S_{\varepsilon(a)b}], \\ S_{AB} &= 2^{s(b)-1} [S_{Ab} + S_{A\varepsilon(b)}]. \end{aligned} \quad (3.7)$$

Since the modular transformation matrix obeys

$$S_{\varepsilon(a)b} = S_{a\varepsilon(b)} \quad (3.8)$$

it follows that

$$\begin{aligned} S_{Ab} &= S_{A\varepsilon(b)}, \\ S_{AB} &= 2^{s(b)} S_{Ab}. \end{aligned} \quad (3.9)$$

### Simple current orbits of $\widehat{\mathfrak{so}}(N)_K$ .

Both  $\widehat{\mathfrak{so}}(2n+1)_K$  and  $\widehat{\mathfrak{so}}(2n)_K$  Dynkin diagrams have a  $\mathbb{Z}_2$ -symmetry that exchanges the 0<sup>th</sup> and 1<sup>st</sup> nodes. This symmetry induces a simple-current symmetry (denoted by  $\sigma$ ) of the  $\widehat{\mathfrak{so}}(N)_K$  WZW model that pairs integrable representations related by  $a_0 \leftrightarrow a_1$ , with the other Dynkin indices unchanged.<sup>8</sup> Their respective Young tableaux are related by  $\ell_1 \rightarrow K - \ell_1$ . Under  $\sigma$ , tensor representations are mapped to tensors, and spinor representations to spinors.

We will refer to representations of  $\widehat{\mathfrak{so}}(N)_K$  with  $\ell_1 < \frac{1}{2}K$ ,  $\ell_1 = \frac{1}{2}K$ , and  $\ell_1 > \frac{1}{2}K$  as being of types I, II, and III respectively. Type II representations are invariant under  $\sigma$ , and are tensors (resp. spinors) when  $K$  is even (resp. odd). Each simple-current orbit of  $\widehat{\mathfrak{so}}(N)_K$  contains either a type I and type III representation, or a single type II representation. We define

$$t(a) = \begin{cases} 1 & \text{if } \ell_1 = \frac{1}{2}K \text{ (type II); that is, } a = \sigma(a), \\ 0 & \text{if } \ell_1 \neq \frac{1}{2}K \text{ (type I or III); that is, } a \neq \sigma(a). \end{cases} \quad (3.10)$$

Finally, the modular transformation matrix of  $\widehat{\mathfrak{so}}(N)_K$  obeys

$$S_{\sigma(a)b} = \pm S_{\varepsilon(a)b} \quad \text{for } b \text{ a } \begin{cases} \text{tensor} \\ \text{spinor} \end{cases} \text{ representation.} \quad (3.11)$$

## 4 Level-rank duality of untwisted D-branes of $\widehat{\mathfrak{so}}(N)_K$

Having reviewed the characterization of integrable representations of  $\widehat{\mathfrak{so}}(N)_K$  in the previous section, we now turn to the untwisted D-branes of the  $\widehat{\mathfrak{so}}(N)_K$  WZW model, which are labelled by those representations. In this section, we will demonstrate a level-rank duality between the untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  and those of  $\widehat{\mathfrak{so}}(K)_N$ .

Since the multiplicities of the representations carried by an open string stretched between two untwisted D-branes are given by the fusion coefficients of the WZW model (2.7), level-rank duality of the untwisted D-branes of the  $\widehat{\mathfrak{so}}(N)_K$  model is closely related to level-rank duality of the fusion coefficients of this model, which was described in ref. [3]. We recall two salient aspects of this duality:

---

<sup>8</sup>Except for  $\widehat{\mathfrak{so}}(4)_K$ , in which case  $\sigma$  acts by  $a_0 \leftrightarrow \min(a_1, a_2)$ . Thus, if  $a$  has Dynkin indices  $(a_0, a_1, a_2)$  then  $\sigma(a)$  has Dynkin indices  $(\min(a_1, a_2), K - a_2, K - a_1)$ .

- The level-rank map is partial: it only relates the *tensor* representations<sup>9</sup> of  $\widehat{\mathfrak{so}}(N)_K$  to those of  $\widehat{\mathfrak{so}}(K)_N$ .
- The level-rank map is not one-to-one between integrable tensor representations  $a$ , but rather between *equivalence classes* of representations,<sup>10</sup> denoted by  $[a]$ . These equivalence classes are characterized by tensor Young tableaux with  $\leq N/2$  rows and  $\leq K/2$  columns (termed “reduced and cominimally-reduced” in ref. [3]). Level-rank duality acts by transposing these tableaux, and maps the set of tensor Young tableaux with  $\leq N/2$  rows and  $\leq K/2$  columns one-to-one onto the set of tensor Young tableaux with  $\leq K/2$  rows and  $\leq N/2$  columns.

The equivalence classes of integrable tensor representations fall into several categories, which we now describe, using the notation of the previous section.

(1)  $s(a) = 0$  and  $t(a) = 1$ : the equivalence class labelled by a tensor Young tableau with  $\ell_1 = \frac{1}{2}K$  columns (only possible when  $K$  is even) and with fewer than  $\frac{1}{2}N$  rows corresponds to a *single* (type II) irreducible representation  $a$ , whose Dynkin indices satisfy  $a_0 = a_1$  and (for  $N = 2n$ )  $a_n = a_{n-1}$ . This representation is invariant under both  $\sigma$  and  $\varepsilon$ .

(2)  $s(a) = 0$  and  $t(a) = 0$ : the equivalence class labelled by a tensor Young tableau with  $\ell_1 < \frac{1}{2}K$  columns and with fewer than  $\frac{1}{2}N$  rows corresponds to a *pair* of irreducible representations  $a$  and  $\sigma(a)$  (of type I and type III) whose Dynkin indices are related by  $a_0 \leftrightarrow a_1$ . When  $N = 2n$ , the Dynkin indices of these representations satisfy  $a_n = a_{n-1}$ , i.e., these representations are invariant under  $\varepsilon$ .

(3)  $s(a) = 1$  and  $t(a) = 1$ : the equivalence class labelled by a tensor Young tableau with  $\ell_1 = \frac{1}{2}K$  columns (only possible when  $K$  is even) and with exactly  $\frac{1}{2}N$  rows (only possible when  $N$  is even) corresponds to a *pair* of (type II) irreducible representations  $a$  and  $\varepsilon(a)$ , whose Dynkin indices are related by  $a_n \leftrightarrow a_{n-1}$  where  $N = 2n$ , and obey<sup>11</sup>  $a_0 = a_1$ , i.e., these representations are invariant under  $\sigma$ .

(4)  $s(a) = 1$  and  $t(a) = 0$ : the equivalence class labelled by a tensor Young tableau with  $\ell_1 < \frac{1}{2}K$  columns and with exactly  $\frac{1}{2}N$  rows (only possible when  $N$  is even) corresponds to *four* irreducible representations:  $a$ ,  $\sigma(a)$ ,  $\varepsilon(a)$ , and  $\varepsilon(\sigma(a))$  (two of type I and two of type III).

Let  $[\tilde{a}]$  denote the transpose of the Young tableau characterizing the equivalence class  $[a]$ . Then

$$t(a) = \tilde{s}(\tilde{a}) \quad \text{and} \quad s(a) = \tilde{t}(\tilde{a}) \quad (4.1)$$

where  $\tilde{s}$  and  $\tilde{t}$  are the quantities (3.6) and (3.10) defined for  $\widehat{\mathfrak{so}}(K)_N$ . Under level-rank duality, equivalence classes  $[a]$  in categories (1), (2), (3), and (4) map into equivalence classes  $[\tilde{a}]$  in categories (4), (2), (3), and (1) respectively.

---

<sup>9</sup> For  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ , a level-rank map between the spinor representations also exists [3], and thus a level-rank map can be defined for all the untwisted D-branes of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ .

<sup>10</sup> This is also the case for level-rank duality of  $\widehat{\mathfrak{su}}(N)_K$ .

<sup>11</sup> For  $\widehat{\mathfrak{so}}(4)_K$ , they obey  $a_0 = \min(a_1, a_2)$ .



We now elucidate the implications of level-rank duality for the untwisted tensor D-branes of the  $\widehat{\mathfrak{so}}(N)_K$  WZW model. Consider the linear combination of untwisted Cardy states

$$|[\alpha]\rangle\rangle = \frac{1}{\sqrt{2}^{t(\alpha)-s(\alpha)+3}} \left[ |\alpha\rangle\rangle_C + |\sigma(\alpha)\rangle\rangle_C + |\varepsilon(\alpha)\rangle\rangle_C + |\varepsilon(\sigma(\alpha))\rangle\rangle_C \right] \quad (4.2)$$

which corresponds to an equivalence class  $[\alpha]$  of integrable tensor representations. Using eqs. (2.7) and (3.11), we find that the multiplicity  $n_{[\beta][\lambda]}^{[\alpha]}$  of the equivalence class of representations  $[\lambda]$  carried by an open string stretched between untwisted D-branes corresponding to the states  $|[\alpha]\rangle\rangle$  and  $|[\beta]\rangle\rangle$  is given by

$$n_{[\beta][\lambda]}^{[\alpha]} = \frac{1}{\sqrt{2}^{t(\alpha)+t(\beta)+t(\lambda)-s(\alpha)-s(\beta)-s(\lambda)+3}} \times \sum_{\substack{\mu=\text{tensor} \\ \text{representations}}} \frac{(S_{\beta\mu} + S_{\varepsilon(\beta)\mu}) (S_{\lambda\mu} + S_{\varepsilon(\lambda)\mu}) (S_{\alpha\mu}^* + S_{\varepsilon(\alpha)\mu}^*)}{S_{0\mu}} \quad (4.3)$$

where only integrable *tensor* representations  $\mu$  remain in the sum as a consequence of eq. (3.11). Using eqs. (3.7) and (3.9), we express  $n_{[\beta][\lambda]}^{[\alpha]}$  in terms of Young tableaux  $A$ ,  $B$ ,  $\Lambda$ ,  $M$ , related to  $\alpha$ ,  $\beta$ ,  $\lambda$ , and  $\mu$  by eq. (3.5),

$$n_{[\beta][\lambda]}^{[\alpha]} = \frac{1}{\sqrt{2}^{t(\alpha)+t(\beta)+t(\lambda)+s(\alpha)+s(\beta)+s(\lambda)-3}} \sum_{\substack{M=\text{tensor} \\ \text{tableaux I,II,III}}} \frac{1}{2^{s(\mu)}} \frac{S_{BM} S_{\Lambda M} S_{AM}^*}{S_{0M}}. \quad (4.4)$$

Finally, since  $S_{A\sigma(M)} = S_{AM}$ , the sum may be restricted to tableaux of types I and II (“cominimally-reduced” tableaux)

$$n_{[\beta][\lambda]}^{[\alpha]} = \frac{1}{\sqrt{2}^{t(\alpha)+t(\beta)+t(\lambda)+s(\alpha)+s(\beta)+s(\lambda)-3}} \sum_{\substack{M=\text{tensor} \\ \text{tableaux I,II}}} \frac{1}{2^{s(\mu)+t(\mu)-1}} \frac{S_{BM} S_{\Lambda M} S_{AM}^*}{S_{0M}}. \quad (4.5)$$

The multiplicities  $n_{[\beta][\lambda]}^{[\alpha]}$  are closely related to  $\Sigma_{B\Lambda}^A$  defined in eq. (3.16) of ref. [3].

Level-rank duality maps the state  $|[\alpha]\rangle\rangle$  of  $\widehat{\mathfrak{so}}(N)_K$  to the state  $|[\tilde{\alpha}]\rangle\rangle$  of  $\widehat{\mathfrak{so}}(K)_N$ . Let  $\tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}^{[\tilde{\alpha}]}$  denote the quantity (4.3) defined for  $\widehat{\mathfrak{so}}(K)_N$ . The form of eq. (4.5) makes manifest the equality of the multiplicities

$$n_{[\beta][\lambda]}^{[\alpha]} = \tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}^{[\tilde{\alpha}]} \quad (4.6)$$

as a consequence of three facts: (1) the set of cominimally-reduced tableaux  $M$  of  $\widehat{\mathfrak{so}}(N)_K$  are in one-to-one correspondence with those of  $\widehat{\mathfrak{so}}(K)_N$ , (2) eq. (4.1) holds for all tensor representations, and (3) the quantities  $S_{AB}$ , defined by eq. (3.7), are level-rank dual ( $S_{AB} = \tilde{S}_{\tilde{A}\tilde{B}}$ ) as was proved in the appendix of ref. [3]. Hence, the spectrum of representations carried by open strings stretched between untwisted tensor D-branes of  $\widehat{\mathfrak{so}}(N)_K$  is level-rank dual.

We end this section by describing the level-rank duality of untwisted spinor D-branes of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ . The equivalence classes  $[\alpha]$  of spinor representations of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  are characterized by type I and type II spinor tableaux, where a type I tableau represents a pair of spinor representations  $\alpha$  and  $\sigma(\alpha)$ , and a type II tableau represents a single irreducible spinor representation  $\alpha$  that obeys  $\sigma(\alpha) = \alpha$ . The level-rank map  $[\alpha] \rightarrow [\hat{\alpha}]$  between equivalence classes of spinor representations of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  and  $\widehat{\mathfrak{so}}(2k+1)_{2n+1}$  was presented in ref. [3]:

- reduce each of the row lengths by  $\frac{1}{2}$ , so that they all become integers
- transpose the resulting tableau
- take the complement with respect to a  $k \times n$  rectangle and rotate 180 degrees
- add  $\frac{1}{2}$  to each of the row lengths.

This takes type I spinor tableaux of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  to type II spinor tableaux of  $\widehat{\mathfrak{so}}(2k+1)_{2n+1}$  and vice versa:  $t(\alpha) = 1 - \tilde{t}(\hat{\alpha})$ . This procedure thus defines a map between an untwisted spinor D-brane of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  corresponding to the boundary state

$$|[\alpha]\rangle\rangle = \frac{1}{\sqrt{2}^{t(\alpha)+1}} \left[ |\alpha\rangle\rangle_C + |\sigma(\alpha)\rangle\rangle_C \right] \quad (4.7)$$

and an untwisted spinor D-brane  $|[\hat{\alpha}]\rangle\rangle$  of  $\widehat{\mathfrak{so}}(2k+1)_{2n+1}$ . The multiplicity of the (equivalence class of) representations  $[\lambda]$  carried by an open string stretched between untwisted spinor D-branes  $[\alpha]$  and  $[\beta]$  of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  obeys

$$n_{[\beta][\lambda]}^{[\alpha]} = \frac{1}{\sqrt{2}^{t(\alpha)+t(\beta)-5}} \sum_{\substack{\mu=\text{tensor} \\ \text{tableaux } 1}} \frac{S_{\beta\mu} S_{\lambda\mu} S_{\alpha\mu}^*}{S_{0\mu}} = \tilde{n}_{[\hat{\beta}][\hat{\lambda}]}^{[\hat{\alpha}]} \quad (4.8)$$

using eq. (3.25) of ref. [3]. Hence, the spectrum of representations carried by open strings stretched between untwisted spinor D-branes of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  is also level-rank dual.

## 5 The $\varepsilon$ -twisted D-branes of the $\widehat{\mathfrak{so}}(2n)_K$ model

In the previous section, we proved the level-rank duality of untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$ . In this section, we will describe a class of twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_K$ , and in the next section, we will prove the level-rank duality of a subset of these twisted D-branes.

Recall from in sec. 3 that the finite Lie algebra  $\mathfrak{so}(2n)$  possesses (when  $n \geq 2$ ) a  $\mathbb{Z}_2$ -automorphism  $\varepsilon$  (chirality flip), under which the Dynkin indices  $a_{n-1}$  and  $a_n$  of an irreducible representation are exchanged. This automorphism lifts to an automorphism of the affine Lie algebra  $\widehat{\mathfrak{so}}(2n)_K$ , and gives rise to a set of  $\varepsilon$ -twisted Ishibashi states and  $\varepsilon$ -twisted Cardy states of the bulk  $\widehat{\mathfrak{so}}(2n)_K$  WZW model, and a corresponding class of  $\varepsilon$ -twisted D-branes of the boundary model. In this section we characterize these twisted states, relying heavily on ref. [20].

**$\varepsilon$ -twisted Ishibashi states of  $\widehat{\mathfrak{so}}(2n)_K$ .**

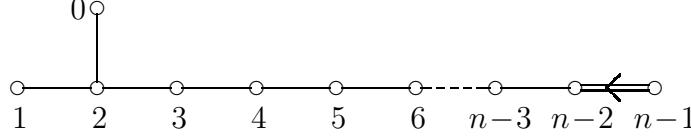
The  $\varepsilon$ -twisted Ishibashi states  $|\mu\rangle\rangle_I^\varepsilon$  of the  $\widehat{\mathfrak{so}}(2n)_K$  WZW model are labelled by integrable representations  $\mu \in \mathcal{E}^\varepsilon$  of  $\widehat{\mathfrak{so}}(2n)_K = (D_n^{(1)})_K$  that obey  $\varepsilon(\mu) = \mu$  (i.e., integrable representations characterized by  $\mathfrak{so}(2n)$  tensor Young tableaux with no more than  $n-1$  rows). These representations have Dynkin indices

$$(\mu_0, \mu_1, \dots, \mu_{n-2}, \mu_{n-1}, \mu_{n-1}) \quad (5.1)$$

that satisfy<sup>12</sup>

$$\mu_0 + \mu_1 + 2(\mu_2 + \cdots + \mu_{n-1}) = K. \quad (5.2)$$

Equivalently, the  $\varepsilon$ -twisted Ishibashi states of  $\widehat{\mathfrak{so}}(2n)_K$  may be characterized by the integrable representations of the associated orbit Lie algebra  $\check{\mathfrak{g}}^\varepsilon = (A_{2n-3}^{(2)})_K$  [20] whose Dynkin diagram is



and whose dual Coxeter numbers are  $(m_0, m_1, m_2, \dots, m_{n-1}) = (1, 1, 2, \dots, 2)$ , where the labelling of nodes is indicated on the diagram. The  $(D_n^{(1)})_K$  representation with Dynkin indices (5.1) corresponds to the  $(A_{2n-3}^{(2)})_K$  representation with Dynkin indices  $(\mu_0, \mu_1, \dots, \mu_{n-2}, \mu_{n-1})$ , whose integrability condition is precisely (5.2).

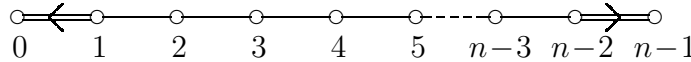
It was shown in ref. [20] that each  $\varepsilon$ -twisted Ishibashi state  $\mu$  of  $\widehat{\mathfrak{so}}(2n)_K$  may be mapped to a *spinor* representation  $\mu'$  of the untwisted affine Lie algebra  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  with Dynkin indices<sup>13</sup>

$$\mu'_i = \mu_i \quad (0 \leq i \leq n-2) \quad \text{and} \quad \mu'_{n-1} = 2\mu_{n-1} + 1. \quad (5.3)$$

The constraint (5.2) on  $\mu$  is precisely equivalent to the integrability constraint (3.1) on the representation  $\mu'$  of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ . Hence,  *$\varepsilon$ -twisted Ishibashi states of  $\widehat{\mathfrak{so}}(2n)_K$  are in one-to-one correspondence with the set of integrable spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  of type I, type II (when  $K$  is even), and type III.*

### $\varepsilon$ -twisted Cardy states of $\widehat{\mathfrak{so}}(2n)_K$ .

The  $\varepsilon$ -twisted Cardy states  $|\alpha\rangle\rangle_C^\varepsilon$  (and therefore the  $\varepsilon$ -twisted D-branes) of the  $\widehat{\mathfrak{so}}(2n)_K$  WZW model<sup>14</sup> are labelled by the integrable representations  $\alpha \in \mathcal{B}^\varepsilon$  of the  $\varepsilon$ -twisted affine Lie algebra  $\hat{\mathfrak{g}}_K^\varepsilon = (D_n^{(2)})_K$  [20], whose Dynkin diagram is



and whose dual Coxeter numbers are  $(m_0, m_1, \dots, m_{n-2}, m_{n-1}) = (1, 2, \dots, 2, 1)$ , where the labelling of nodes is indicated on the diagram. The Dynkin indices  $(\alpha_0, \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1})$  of  $\alpha$  thus satisfy<sup>15</sup>

$$\alpha_0 + 2(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1} = K. \quad (5.4)$$

It was shown in ref. [20] that each  $\varepsilon$ -twisted Cardy state  $\alpha$  of  $\widehat{\mathfrak{so}}(2n)_K$  may be mapped to a representation  $\alpha'$  of the untwisted affine Lie algebra  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  with Dynkin indices<sup>16</sup>

$$\alpha'_0 = \alpha_0 + \alpha_1 + 1 \quad \text{and} \quad \alpha'_i = \alpha_i \quad (1 \leq i \leq n-1). \quad (5.5)$$

<sup>12</sup>For  $\widehat{\mathfrak{so}}(4)_K$ , the Dynkin indices  $(\mu_0, \mu_1, \mu_2)$  satisfy  $\mu_0 + \mu_1 = K$ .

<sup>13</sup>Except for  $\widehat{\mathfrak{so}}(4)_K$ , in which case  $\mu'_0 = 2\mu_0 + 1$  and  $\mu'_1 = 2\mu_1 + 1$ . Also, recall footnote 6.

<sup>14</sup>  $n \geq 2$  is understood.

<sup>15</sup>For  $\widehat{\mathfrak{so}}(4)_K$ , the condition is  $\alpha_0 + \alpha_1 = K$ .

<sup>16</sup>Except for  $\widehat{\mathfrak{so}}(4)_K$ , in which case  $\alpha'_0 = 2\alpha_0 + \alpha_1 + 2$  and  $\alpha'_1 = \alpha_1$ . Also, recall footnote 6.

The constraint (5.4) on  $\alpha$  implies that  $\alpha'$  is an integrable representation of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ , and the constraint  $\alpha_0 \geq 0$  further implies that  $\alpha'$  is a type I representation of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  (i.e., corresponds to a Young tableau whose first row length obeys  $\ell'_1 \leq \frac{1}{2}K$ ). Therefore,  *$\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_K$  are in one-to-one correspondence with the set of integrable type I representations of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ , of both tensor and spinor types.*

Although the  $\varepsilon$ -twisted Ishibashi states and the  $\varepsilon$ -twisted Cardy states of  $\widehat{\mathfrak{so}}(2n)_K$  are characterized differently in terms of integrable representations of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ , they are equal in number. (For  $K$  even, the number of each is  $\binom{n-1+K/2}{n-1} + \binom{n-2+K/2}{n-1}$ , while for  $K$  odd, the number of each is  $2 \binom{n-1+(K-1)/2}{n-1}$ .) Thus, the  $\varepsilon$ -twisted Cardy states  $\alpha$  may be written as linear combinations (2.4) of  $\varepsilon$ -twisted Ishibashi states  $\mu$ , with the transformation coefficients  $\psi_{\alpha\mu}$  given by the modular transformation matrix of  $(D_n^{(2)})_K$ . In ref. [20], it was shown that these coefficients are proportional to the (real) matrix elements  $S'_{\alpha'\mu'}$  of the modular transformation matrix of the untwisted affine Lie algebra  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ :

$$\psi_{\alpha\mu} = \sqrt{2} S'_{\alpha'\mu'} = \sqrt{2} S'^*_{\alpha'\mu'} \quad (5.6)$$

where  $\alpha'$  and  $\mu'$  are the  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  representations related to  $\alpha$  and  $\mu$  by eqs. (5.5) and (5.3) respectively.

### Twisted open string partition function of $\widehat{\mathfrak{so}}(2n)_K$ .

Combining eqs. (2.6) and (5.6), we may write the multiplicities of the representations carried by an open string stretched between  $\varepsilon$ -twisted D-branes  $\alpha$  and  $\beta$  of  $\widehat{\mathfrak{so}}(2n)_K$  as

$$n_{\beta\lambda}^\alpha = \sum_{\mu'=\text{spinors I,II,III}} \frac{2 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} \quad (5.7)$$

where  $S_{\lambda\mu}$  and  $S'_{\alpha'\mu'}$  are modular transformation matrix elements of  $\widehat{\mathfrak{so}}(2n)_K$  and  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  respectively, and the sum is over all  $\varepsilon$ -twisted Ishibashi states  $\mu$  of  $\widehat{\mathfrak{so}}(2n)_K$ , or equivalently, over all integrable spinor representations  $\mu'$  of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ . (Type II spinors of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  are present only when  $K$  is even.)

Although the  $\varepsilon$ -twisted D-branes correspond to both tensor and spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ , for the remainder of this section we will restrict  $\alpha$  and  $\beta$  to correspond to spinor representations  $\alpha'$  and  $\beta'$  of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ , which allows us to simplify eq. (5.7) considerably. Recall from eq. (3.11) that the modular transformation matrix elements of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  obey

$$S'_{\alpha'\sigma(\mu')} = -S'_{\alpha'\mu'} \quad \text{for } \alpha' = \text{spinor}. \quad (5.8)$$

As a consequence, type I and III representations  $\mu'$ , which are related by  $\sigma$ , may be combined in eq. (5.7)

$$n_{\beta\lambda}^\alpha = \sum_{\mu'=\text{spinors I}} \left[ \frac{2 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} + \frac{2 S'_{\alpha'\mu'} S_{\lambda\sigma(\mu)} S'_{\beta'\mu'}}{S_{0\sigma(\mu)}} \right] \quad \text{for } \alpha', \beta' \text{ both spinors} \quad (5.9)$$

and type II representations, which obey  $\sigma(\mu') = \mu'$ , drop out of the sum since  $S'_{\alpha'\mu'} = 0$ . (We have also used the fact that the  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$  representation  $\sigma(\mu')$ , related to  $\mu'$  by  $\ell'_1 \rightarrow K+1-\ell'_1$ , corresponds via the map (5.3) to the  $\varepsilon$ -twisted Ishibashi state  $\sigma(\mu)$  of

$\widehat{\mathfrak{so}}(2n)_K$ , related to  $\mu$  by  $\ell_1 \rightarrow K - \ell_1$ .) Finally, recalling that the modular transformation matrix elements of  $\widehat{\mathfrak{so}}(2n)_K$  obey

$$S_{\lambda\sigma(\mu)} = \pm S_{\lambda\varepsilon(\mu)} \quad \text{for } \lambda \text{ a } \begin{cases} \text{tensor} \\ \text{spinor} \end{cases} \text{ representation} \quad (5.10)$$

and that  $\varepsilon$ -twisted Ishibashi states obey  $\varepsilon(\mu) = \mu$ , we finally obtain

$$n_{\beta\lambda}{}^\alpha = \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} \quad \begin{array}{l} \text{for } \alpha', \beta' \text{ both spinors} \\ \text{and } \lambda = \text{tensor} \end{array} \quad (5.11)$$

and

$$n_{\beta\lambda}{}^\alpha = 0 \quad \begin{array}{l} \text{for } \alpha', \beta' \text{ both spinors} \\ \text{and } \lambda = \text{spinor.} \end{array} \quad (5.12)$$

This result will allow us to demonstrate in the next section the level-rank duality of the spectrum of an open string stretched between spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$ .

## 6 Level-rank duality of $\varepsilon$ -twisted D-branes of $\widehat{\mathfrak{so}}(2n)_{2k}$

As we saw in the previous section, the  $\widehat{\mathfrak{so}}(2n)_K$  WZW model possesses twisted D-branes corresponding to the chirality-flip symmetry  $\varepsilon$  of the  $\mathfrak{so}(2n)$  Dynkin diagram, and these  $\varepsilon$ -twisted D-branes are characterized by integrable type I tensor and spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{K+1}$ . We will refer to these as tensor and spinor  $\varepsilon$ -twisted D-branes respectively.

In this section, we will exhibit a level-rank duality<sup>17</sup> between the  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  and those of  $\widehat{\mathfrak{so}}(2k)_{2n}$ . This duality is partial, and only holds between *spinor*  $\varepsilon$ -twisted D-branes (just as the level-rank duality of untwisted D-branes only holds between tensor D-branes). The restriction to spinor  $\varepsilon$ -twisted D-branes can be anticipated by observing that the number of tensor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  is  $\binom{n+k-1}{n-1}$  and the number of spinor  $\varepsilon$ -twisted D-branes is  $\binom{n+k-2}{n-1}$ , and only the latter is invariant under  $n \leftrightarrow k$ .

First we define an explicit one-to-one map  $\alpha \rightarrow \hat{\alpha}$  between the spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  and  $\widehat{\mathfrak{so}}(2k)_{2n}$ . The map  $\alpha \rightarrow \hat{\alpha}$  is defined by specifying its action<sup>18</sup> on the corresponding  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$  and  $\widehat{\mathfrak{so}}(2k-1)_{2n+1}$  representations  $\alpha'$  and  $\hat{\alpha}'$ :

- reduce each of the row lengths of  $\alpha'$  by  $\frac{1}{2}$ , so that they all become integers
- transpose the resulting tableau
- add  $\frac{1}{2}$  to each of the row lengths.

<sup>17</sup> Clearly the  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k+1}$  have no level-rank duals, since  $\widehat{\mathfrak{so}}(2k+1)_{2n}$  has no  $\varepsilon$ -twisted D-branes.

<sup>18</sup> Note that the “hat” map defined here differs from that defined in sec. 4 between spinor representations of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$  and  $\widehat{\mathfrak{so}}(2k+1)_{2n+1}$ . The map defined here also characterizes the map between  $\omega_c$ -twisted D-branes of  $\widehat{\mathfrak{su}}(2n+1)_{2k+1}$  and  $\widehat{\mathfrak{su}}(2k+1)_{2n+1}$  [7].

The same procedure defines a one-to-one map  $\mu' \rightarrow \hat{\mu}'$  between type I spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$  and  $\widehat{\mathfrak{so}}(2k-1)_{2n+1}$  corresponding to  $\varepsilon$ -twisted Ishibashi states. By virtue of eq. (5.3), this map lifts to a map  $\mu \rightarrow \tilde{\mu}$  between (a subset of) the  $\varepsilon$ -twisted Ishibashi states. As suggested by the notation, this map is simply transposition of the type I  $\widehat{\mathfrak{so}}(2n)_{2k}$  Young tableau corresponding to  $\mu$ .

Next, we turn to the level-rank duality of the spectrum of an open string stretched between  $\varepsilon$ -twisted D-branes. In the previous section, it was shown that the multiplicity of the representation  $\lambda$  carried by an open string stretched between spinor  $\varepsilon$ -twisted D-branes  $\alpha$  and  $\beta$  of  $\widehat{\mathfrak{so}}(2n)_{2k}$  is given by

$$n_{\beta\lambda}{}^\alpha = \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} S_{\lambda\mu} S'_{\beta'\mu'}}{S_{0\mu}} \quad \text{for } \lambda = \text{tensor} \quad (6.1)$$

with  $n_{\beta\lambda}{}^\alpha$  vanishing for  $\lambda = \text{spinor}$ . As in sec. 4, however, we consider the multiplicity corresponding to the equivalence class of tensor representations  $[\lambda]$ :

$$n_{\beta[\lambda]}{}^\alpha = \frac{1}{\sqrt{2}^{t(\lambda)-s(\lambda)+3}} \left[ n_{\beta\lambda}{}^\alpha + n_{\beta\varepsilon(\lambda)}{}^\alpha + n_{\beta\sigma(\lambda)}{}^\alpha + n_{\beta\varepsilon(\sigma(\lambda))}{}^\alpha \right]. \quad (6.2)$$

Using eqs. (6.1), (3.11), and (3.7), we find

$$\begin{aligned} n_{\beta[\lambda]}{}^\alpha &= \frac{1}{\sqrt{2}^{t(\lambda)-s(\lambda)+1}} \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} (S_{\lambda\mu} + S_{\varepsilon(\lambda)\mu}) S'_{\beta'\mu'}}{S_{0\mu}} \\ &= \frac{1}{\sqrt{2}^{t(\lambda)+s(\lambda)-1}} \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} S_{\Lambda M} S'_{\beta'\mu'}}{S_{0M}} \end{aligned} \quad (6.3)$$

where  $\Lambda = 2^{s(\lambda)-1} [\lambda \oplus \varepsilon(\lambda)]$  and  $M = \mu$  since  $\varepsilon(\mu) = \mu$ . The form of eq. (6.3) makes manifest the equality of the multiplicities

$$n_{\beta[\lambda]}{}^\alpha = \frac{1}{\sqrt{2}^{t(\lambda)+s(\lambda)-1}} \sum_{\mu'=\text{spinors I}} \frac{4 S'_{\alpha'\mu'} S_{\Lambda M} S'_{\beta'\mu'}}{S_{0M}} \quad (6.4)$$

$$= \frac{1}{\sqrt{2}^{\tilde{s}(\tilde{\lambda})+\tilde{t}(\tilde{\lambda})-1}} \sum_{\hat{\mu}'=\text{spinors I}} \frac{4 \tilde{S}'_{\alpha'\hat{\mu}'} \tilde{S}_{\tilde{\Lambda}\tilde{M}} \tilde{S}'_{\hat{\beta}'\hat{\mu}'}}{\tilde{S}_{0\tilde{M}}} \quad (6.5)$$

$$= \tilde{n}_{\hat{\beta}[\tilde{\lambda}]}{}^{\hat{\alpha}} \quad (6.6)$$

where we have used eq. (4.1) and the facts that:

- (1) type I spinors  $\mu'$  of  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$  map one-to-one to type I spinors  $\hat{\mu}'$  of  $\widehat{\mathfrak{so}}(2k-1)_{2n+1}$ ,
- (2)  $S_{\Lambda M} = \tilde{S}_{\tilde{\Lambda}\tilde{M}}$  [3], where  $S$  and  $\tilde{S}$  are the modular transformation matrices of  $\widehat{\mathfrak{so}}(2n)_{2k}$  and  $\widehat{\mathfrak{so}}(2k)_{2n}$  respectively, and
- (3)  $S'_{\alpha'\mu'} = \tilde{S}'_{\hat{\alpha}'\hat{\mu}'}$  for  $\alpha'$  and  $\mu'$  both type I spinor representations (eq. (6.10) of ref. [7]), where  $S'$  and  $\tilde{S}'$  are the modular transformation matrices of  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$  and  $\widehat{\mathfrak{so}}(2k-1)_{2n+1}$  respectively. Since by eq. (5.12) only tensor representations  $\lambda$  appear in the  $\varepsilon$ -twisted open-string partition function (2.5), we have established that the spectrum of representations carried by open strings stretched between  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  is level-rank dual.

## 7 Conclusions

We have analyzed the level-rank duality of the untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  and of the  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$ . In each case, only a subset of the D-branes are mapped onto those of the level-rank-dual theory.

Untwisted D-branes of  $\widehat{\mathfrak{so}}(N)_K$  are characterized by integrable tensor and spinor representations of  $\widehat{\mathfrak{so}}(N)_K$ . Only the untwisted *tensor* D-branes participate in level-rank duality.<sup>19</sup> The tensor representations  $\alpha$  of  $\widehat{\mathfrak{so}}(N)_K$  fall into equivalence classes  $[\alpha]$  generated by the  $\mathbb{Z}_2$ -isomorphisms  $\sigma$  and  $\varepsilon$  (the latter non-trivial only for  $N$  even), and characterized by Young tableaux with  $\leq N/2$  rows and  $\leq K/2$  columns. Level-rank duality acts by transposing these tableaux, and thus maps the equivalence classes  $[\alpha]$  of untwisted tensor D-branes of  $\widehat{\mathfrak{so}}(N)_K$  onto  $[\tilde{\alpha}]$  of  $\widehat{\mathfrak{so}}(K)_N$ . We showed that the multiplicity  $n_{[\beta][\lambda]}^{[\alpha]}$  of the (equivalence class of) representations  $[\lambda]$  carried by an open string stretched between untwisted  $\widehat{\mathfrak{so}}(N)_K$  D-branes corresponding to  $[\alpha]$  and  $[\beta]$  is equal to  $\tilde{n}_{[\tilde{\beta}][\tilde{\lambda}]}^{[\tilde{\alpha}]}$ , the multiplicity of the (equivalence class of) representations  $[\tilde{\lambda}]$  carried by an open string stretched between untwisted  $\widehat{\mathfrak{so}}(K)_N$  D-branes corresponding to  $[\tilde{\alpha}]$  and  $[\tilde{\beta}]$ . A similar result was shown for untwisted spinor D-branes of  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ .

The  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$ , associated with the chirality-flip symmetry  $\varepsilon$  of the  $\mathfrak{so}(2n)$  Dynkin diagram, are characterized by type I integrable tensor and spinor representations of  $\widehat{\mathfrak{so}}(2n-1)_{2k+1}$ . Only the *spinor*  $\varepsilon$ -twisted D-branes participate in level-rank duality. We defined a one-to-one map  $\alpha \rightarrow \hat{\alpha}$  between the spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2n)_{2k}$  and the spinor  $\varepsilon$ -twisted D-branes of  $\widehat{\mathfrak{so}}(2k)_{2n}$ . We then showed that the multiplicity  $n_{\beta[\lambda]}^{\alpha}$  of the (equivalence class of) representations  $[\lambda]$  carried by an open string stretched between  $\varepsilon$ -twisted  $\widehat{\mathfrak{so}}(2n)_{2k}$  D-branes corresponding to  $\alpha$  and  $\beta$  is equal to  $\tilde{n}_{\tilde{\beta}[\tilde{\lambda}]}^{\hat{\alpha}}$ , the multiplicity of the (equivalence class of) representations  $[\tilde{\lambda}]$  carried by an open string stretched between  $\varepsilon$ -twisted  $\widehat{\mathfrak{so}}(2k)_{2n}$  D-branes corresponding to  $\hat{\alpha}$  and  $\hat{\beta}$ .

Hence, for both untwisted and  $\varepsilon$ -twisted D-branes, we have established an isomorphism between the spectrum of an open string ending on these D-branes and the spectrum of an open string ending on the level-rank-dual D-branes.

In both the  $\widehat{\mathfrak{su}}(N)_K$  and  $\widehat{\mathfrak{sp}}(n)_k$  WZW theories, the charges of level-rank-dual untwisted D-branes are equal (modulo sign) [5, 6], with a slightly more complicated relationship holding between the charges of twisted D-branes [7]. In the case of  $\widehat{\mathfrak{so}}(N)_K$ , however, the charges of the D-branes do not exhibit any simple relationship under level-rank duality.

## Acknowledgments

SN wishes to thank Howard Schnitzer for his collaboration on a long series of papers on which the results of this paper depend.

---

<sup>19</sup> Except for  $\widehat{\mathfrak{so}}(2n+1)_{2k+1}$ , where the untwisted spinor D-branes also respect level-rank duality.

## References

- [1] S. G. Naculich and H. J. Schnitzer, “Duality between  $SU(N)_k$  and  $SU(k)_N$  WZW models.” Nucl. Phys. **B347** (1990) 687–742; S. G. Naculich and H. J. Schnitzer, “Duality relations between  $SU(N)_k$  and  $SU(k)_N$  WZW models and their braid matrices.” Phys. Lett. **B244** (1990) 235–240; M. A. Walton, “Conformal branching rules and modular invariants.” Nucl. Phys. **B322** (1989) 775; D. Altschuler, M. Bauer, and C. Itzykson, “The branching rules of conformal embeddings.” Commun. Math. Phys. **132** (1990) 349–364; J. Fuchs and P. van Driel, “Some symmetries of quantum dimensions.” J. Math. Phys. **31** (1990) 1770–1775; A. Kuniba and T. Nakanishi, “Level rank duality in fusion RSOS models.” In *Proceedings of the International Colloquium on Modern Quantum Field Theory*, Bombay, January 1990 (World Scientific, Singapore, 1991); H. Saleur and D. Altschuler, “Level rank duality in quantum groups.” Nucl. Phys. **B354** (1991) 579–613; T. Nakanishi and A. Tsuchiya, “Level rank duality of WZW models in conformal field theory.” Commun. Math. Phys. **144** (1992) 351–372.
- [2] S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, “Group level duality in WZW models and Chern-Simons theory.” Phys. Lett. **B246** (1990) 417–422.
- [3] E. J. Mlawer, S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, “Group level duality of WZW fusion coefficients and Chern-Simons link observables.” Nucl. Phys. **B352** (1991) 863–896.
- [4] S. G. Naculich and H. J. Schnitzer, “Level-rank duality of the  $U(N)$  WZW model, Chern-Simons theory, and 2d qYM theory.” JHEP **06** (2007) 023, [hep-th/0703089](#).
- [5] S. G. Naculich and H. J. Schnitzer, “Level-rank duality of D-branes on the  $SU(N)$  group manifold.” Nucl. Phys. **B740** (2006) 181–194, [hep-th/0511083](#).
- [6] S. G. Naculich and H. J. Schnitzer, “Level-rank duality of untwisted and twisted D-branes.” Nucl. Phys. **B742** (2006) 295–311, [hep-th/0601175](#).
- [7] S. G. Naculich and H. J. Schnitzer, “Twisted D-branes of the  $SU(N)_K$  WZW model and level-rank duality.” Nucl. Phys. **B755** (2006) 164–185, [hep-th/0606147](#).
- [8] C. Klimcik and P. Severa, “Open strings and D-branes in WZNW models.” Nucl. Phys. **B488** (1997) 653–676, [hep-th/9609112](#); M. Kato and T. Okada, “D-branes on group manifolds.” Nucl. Phys. **B499** (1997) 583–595, [hep-th/9612148](#); A. Y. Alekseev and V. Schomerus, “D-branes in the WZW model.” Phys. Rev. **D60** (1999) 061901, [hep-th/9812193](#); K. Gawedzki, “Conformal field theory: A case study.” [hep-th/9904145](#).
- [9] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, “On the classification of bulk and boundary conformal field theories.” Phys. Lett. **B444** (1998) 163–166, [hep-th/9809097](#); J. Fuchs and C. Schweigert, “Symmetry breaking boundaries. I: General theory.” Nucl. Phys. **B558** (1999) 419–483, [hep-th/9902132](#). R. E. Behrend,



- P. A. Pearce, V. B. Petkova, and J.-B. Zuber, “Boundary conditions in rational conformal field theories.” Nucl. Phys. **B570** (2000) 525–589, [hep-th/9908036](#).
- [10] L. Birke, J. Fuchs, and C. Schweigert, “Symmetry breaking boundary conditions and WZW orbifolds.” Adv. Theor. Math. Phys. **3** (1999) 671–726, [hep-th/9905038](#).
- [11] A. Y. Alekseev, A. Recknagel, and V. Schomerus, “Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres.” JHEP **09** (1999) 023, [hep-th/9908040](#); “Brane dynamics in background fluxes and non-commutative geometry.” JHEP **05** (2000) 010, [hep-th/0003187](#); “Open strings and non-commutative geometry of branes on group manifolds.” Mod. Phys. Lett. **A16** (2001) 325–336, [hep-th/0104054](#); A. Alekseev and V. Schomerus, “RR charges of D2-branes in the WZW model.” [hep-th/0007096](#).
- [12] G. Felder, J. Frohlich, J. Fuchs, and C. Schweigert, “The geometry of WZW branes.” J. Geom. Phys. **34** (2000) 162–190, [hep-th/9909030](#).
- [13] S. Stanciu, “D-branes in group manifolds.” JHEP **01** (2000) 025, [hep-th/9909163](#); “A note on D-branes in group manifolds: Flux quantization and D0-charge.” JHEP **10** (2000) 015, [hep-th/0006145](#); “An illustrated guide to D-branes in SU(3).” [hep-th/0111221](#); J. M. Figueroa-O’Farrill and S. Stanciu, “D-brane charge, flux quantization and relative (co)homology.” JHEP **01** (2001) 006, [hep-th/0008038](#).
- [14] C. Bachas, M. R. Douglas, and C. Schweigert, “Flux stabilization of D-branes.” JHEP **05** (2000) 048, [hep-th/0003037](#); J. Pawelczyk, “SU(2) WZW D-branes and their noncommutative geometry from DBI action.” JHEP **08** (2000) 006, [hep-th/0003057](#); W. Taylor, “D2-branes in B fields.” JHEP **07** (2000) 039, [hep-th/0004141](#).
- [15] S. Fredenhagen and V. Schomerus, “Branes on group manifolds, gluon condensates, and twisted K-theory.” JHEP **04** (2001) 007, [hep-th/0012164](#).
- [16] J. M. Maldacena, G. W. Moore, and N. Seiberg, “Geometrical interpretation of D-branes in gauged WZW models.” JHEP **07** (2001) 046, [hep-th/0105038](#); J. M. Maldacena, G. W. Moore, and N. Seiberg, “D-brane instantons and K-theory charges.” JHEP **11** (2001) 062, [hep-th/0108100](#).
- [17] K. Gawedzki, “Boundary WZW, G/H, G/G and CS theories.” Annales Henri Poincare **3** (2002) 847–881, [hep-th/0108044](#); K. Gawedzki and N. Reis, “WZW branes and gerbes.” Rev. Math. Phys. **14** (2002) 1281–1334, [hep-th/0205233](#).
- [18] H. Ishikawa, “Boundary states in coset conformal field theories.” Nucl. Phys. **B629** (2002) 209–232, [hep-th/0111230](#).
- [19] V. B. Petkova and J. B. Zuber, “Boundary conditions in charge conjugate  $sl(N)$  WZW theories.” [hep-th/0201239](#).
- [20] M. R. Gaberdiel and T. Gannon, “Boundary states for WZW models.” Nucl. Phys. **B639** (2002) 471–501, [hep-th/0202067](#).

- [21] M. R. Gaberdiel and T. Gannon, “The charges of a twisted brane.” JHEP **01** (2004) 018, [hep-th/0311242](#); M. R. Gaberdiel, T. Gannon, and D. Roggenkamp, “The D-branes of  $SU(n)$ .” JHEP **07** (2004) 015, [hep-th/0403271](#); “The coset D-branes of  $SU(n)$ .” JHEP **10** (2004) 047, [hep-th/0404112](#).
- [22] A. Y. Alekseev, S. Fredenhagen, T. Quella, and V. Schomerus, “Non-commutative gauge theory of twisted D-branes.” Nucl. Phys. **B646** (2002) 127–157, [hep-th/0205123](#); T. Quella, “Branching rules of semi-simple Lie algebras using affine extensions.” J. Phys. **A35** (2002) 3743–3754, [math-ph/0111020](#); T. Quella and V. Schomerus, “Symmetry breaking boundary states and defect lines.” JHEP **06** (2002) 028, [hep-th/0203161](#); T. Quella, “On the hierarchy of symmetry breaking D-branes in group manifolds.” JHEP **12** (2002) 009, [hep-th/0209157](#).
- [23] P. Bouwknegt, P. Dawson, and D. Ridout, “D-branes on group manifolds and fusion rings.” JHEP **12** (2002) 065, [hep-th/0210302](#); P. Bouwknegt and D. Ridout, “A note on the equality of algebraic and geometric D-brane charges in WZW models.” JHEP **05** (2004) 029, [hep-th/0312259](#).
- [24] H. Ishikawa and T. Tani, “Novel construction of boundary states in coset conformal field theories.” Nucl. Phys. **B649** (2003) 205–242, [hep-th/0207177](#); H. Ishikawa and A. Yamaguchi, “Twisted boundary states in  $c = 1$  coset conformal field theories.” JHEP **04** (2003) 026, [hep-th/0301040](#); H. Ishikawa and T. Tani, “Twisted boundary states in Kazama-Suzuki models.” Nucl. Phys. **B678** (2004) 363–397, [hep-th/0306227](#); “Twisted boundary states and representation of generalized fusion algebra.” [hep-th/0510242](#); S. Schafer-Nameki, “D-branes in  $N = 2$  coset models and twisted equivariant K- theory.” [hep-th/0308058](#); V. Braun and S. Schafer-Nameki, “Supersymmetric WZW models and twisted K-theory of  $SO(3)$ .” [hep-th/0403287](#).
- [25] M. R. Gaberdiel and T. Gannon, “D-brane charges on non-simply connected groups.” JHEP **04** (2004) 030, [hep-th/0403011](#); S. Fredenhagen, “D-brane charges on  $SO(3)$ .” JHEP **11** (2004) 082, [hep-th/0404017](#); M. Vasudevan, “Charges of exceptionally twisted branes.” JHEP **07** (2005) 035, [hep-th/0504006](#); S. Fredenhagen, M. R. Gaberdiel, and T. Mettler, “Charges of twisted branes: The exceptional cases.” JHEP **05** (2005) 058, [hep-th/0504007](#); S. Fredenhagen and T. Quella, “Generalised permutation branes.” JHEP **11** (2005) 004, [hep-th/0509153](#).
- [26] N. Ishibashi, “The boundary and crosscap states in conformal field theories.” Mod. Phys. Lett. **A4** (1989) 251.
- [27] J. Fuchs, B. Schellekens, and C. Schweigert, “From Dynkin diagram symmetries to fixed point structures.” Commun. Math. Phys. **180** (1996) 39–98, [hep-th/9506135](#).
- [28] J. L. Cardy, “Boundary conditions, fusion rules and the Verlinde formula.” Nucl. Phys. **B324** (1989) 581.
- [29] E. P. Verlinde, “Fusion rules and modular transformations in 2-d conformal field theory.” Nucl. Phys. **B300** (1988) 360.